Answers to Modal Logic Exercises Formal Methods, 2017-04-24

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Page 55

2. **(a)** Reading A:
   (i) Every mathematician is such that they are necessarily: rational and possibly not: bipedal.
   (ii) Every cyclist is such that they are necessarily: bipedal and possibly not: rational.
   (iii) P. K. Z. is a mathematician and a cyclist.
   (iv) Therefore, P. K. Z. does not exist.

   Valid: PKZ would be necessarily rational and possibly not rational, contradiction, so there is no such PKZ. But unsound.

   (b) Reading B:
   (i) Necessarily: everything is such that if it is a mathematician, it is rational; and possibly not: everything is such that if it is a mathematical, it is bipedal.
   (ii) Necessarily: everything is such that if it is a cyclist, it is bipedal; and possibly not: everything is such that if it is a cyclist, it is rational.
   (iii) P. K. Z. is a mathematician and a cyclist.
   (iv) Therefore, P. K. Z. does not exist.

   Invalid: all that follows is that PKZ is both rational and bipedal, not that PKZ is necessarily or contingently either.

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Page 59

2. Suppose a model with two worlds, w and w', where $\mathscr{R} = \{\langle w, w' \rangle\} \neq \varnothing$. At w', $\Box A$ is degenerately true, since that world accesses no world. But at w', $\Diamond A$ is false for the same reason. Since the antecedent is true but the consequent false, $\Box A \supset \Diamond A$ is false at w' and hence this model is a counterexample to that sentence.

3. **(a)** Suppose there is a $\mathscr{L}_\mathsf{K}$ model in which $\Box(A \supset B) \supset (\Box A \supset \Box B)$ is false at w. Then (i) $v(w, \Box(A \supset B)) = 1$ and (ii) $v(w, \Box A) = 1$ and (iii) $v(w, \Box B) = 0$. By claim (iii), we know there must be some w' such that $w\mathscr{R}w'$, and such that $v(w', B) = 0$. By claim (ii), $v(w', A) = 1$. And by claim (i), $v(w', A \supset B) = 1$. But then $v(w', B) = 1$, contradiction. So there is no such w, and distribution of $\Box$ over $\supset$ is a theorem of $\mathscr{L}_\mathsf{K}$.

   (c) Consider the $\mathscr{L}_\mathsf{K}$ model with two worlds w, w'. Let $\mathscr{R} = \{\langle w, w' \rangle, \langle w', w \rangle\}$. Let $v(w, A) = v(w, B) = 0$ and $v(w', A) = v(w', B) = 1$. Since all worlds accessible from w make B true, $v(w, \Box B) = 1$. Since at least one world accessible from w makes A true, $v(w, \Diamond A) = 1$. So $v(w, \Diamond A \wedge \Box B) = 1$. But since at least one world accessible from w' makes B false, $v(w', \Box B) = 0$. So $v(w', A \wedge \Box B) = 0$. Since w' is the only world accessible from w, $v(w, \Diamond(A \wedge \Box B) = 0$. So $v(w, (\Diamond A \wedge \Box B) \supset \Diamond(A \wedge \Box B)) = 0$ – counterexample.

   (e) Consider the two-world $\mathscr{L}_\mathsf{K}$ model with a universal accessibility relation on the domain $\{w, w'\}$, such that $v(w, A) = 0$ and $v(w', A) = 1$. Since each world can access every world, each world can access at least one world where A. Since $\Diamond A$ is true at every world, and w can access every world, so $\Box\Diamond A$ is true at w. But since $v(w, A) = 0$, $v(w, \Box\Diamond A \supset A) = 0$ – counterexample.

Page 67

1. An extension of $\mathscr{L}_\mathsf{K}$ is any language all of whose models are models of $\mathscr{L}_\mathsf{K}$ (p. 60). Since we just showed (ex. 3a from page 59) that $\Box(A \supset B) \supset (\Box A \supset \Box B)$ is valid in $\mathscr{L}_\mathsf{K}$, it remains valid in every extension of $\mathscr{L}_\mathsf{K}$.

2. $\Box(\phi \wedge \psi)$ is logically equivalent to $\Box\phi \wedge \Box\psi$ in $\mathscr{L}_\mathsf{K}$ and all languages extending it. $\Box(p \wedge \neg p)$ (and so the equivalent $\Box p \wedge \Box\neg p$) is unsatisfiable in any

language extending $\mathscr{L}_{\mathsf{K}}^{ser}$ – which is every language canvassed except $\mathscr{L}_{\mathsf{K}}$ – since it can only be satisfied at a world if that world accesses no world. A serial $\mathscr{R}$ guarantees that every world accesses some world.

3. $\Diamond(p \vee \neg p)$ is contingent iff it is satisfiable but not valid. It is satisfiable in $\mathscr{L}_{\mathsf{K}}$, but not valid since a model in which some world accesses no world will make all $\Diamond$ sentences invalid, since all false at that world. But in any language extending $\mathscr{L}_{\mathsf{K}}^{ser}$, in models for which each world accesses some world, $\Diamond(p \vee \neg p)$ is valid because $p \vee \neg p$ is true at every world.

4. The counterexample they offer to $\Diamond A \supset \Box\Diamond A$ involves two worlds w, w' such that $w\mathscr{R}w'$ and each access themselves, where $v(w, A) = 1$ and $v(w', A) = 0$. If we add another world u such that $u\mathscr{R}w$ and $u\mathscr{R}w'$ (and which accesses itself), this is a three world $\mathscr{L}_{\mathsf{K}}^{rt}$ model. But since neither w nor w' access u, the introduction of u doesn't change the truth value of any of the sentences at w or w' – so this model is still a counterexample, because $v(w', \Diamond A) = 0$ so $v(w, \Box\Diamond A) = 0$ while $v(w, \Diamond A) = 1$.

Page 81

1. **(a)** $\Diamond(p \vee q) \rightarrow (\Diamond p \vee \Diamond q)$ is a theorem of K and all stronger tableaux systems.

   (c) $\neg\Diamond\neg p \rightarrow (q \vee \Box p)$ is a theorem of K and all stronger tableaux systems.

   (e) $\Diamond(p \rightarrow q) \leftrightarrow (\Box p \rightarrow \Diamond q)$ is a theorem of K and all stronger tableaux systems.

   (g) $\Diamond p \rightarrow \Diamond\Diamond p$ is a theorem of T and all stronger tableaux systems.

   (i) $(\Box p \wedge (p \rightarrow q)) \rightarrow q$ is a theorem of T and all stronger tableaux systems.

   (k) $(\Box p \wedge p) \rightarrow \Diamond p$ is a theorem of D and all stronger tableaux systems.

3. **(b)** Derivable in K.

   (d) Derivable in K.

   (f) Derivable in S4, but can be counterexampled in T.

   (h) Derivable in S4 (indeed, even in K4).

   (j) Actually not even derivable in S5, though $\Diamond\Box A \supset A$ is provable in S5 (and any language with a symmetric accessibility relation).

5. **(a)** Two sentences are strictly equivalent iff they are necessarily equivalent, i.e., iff they are true at exactly the same worlds. Consider an n-world S5 model; each world has a unique valuation on sentence letters. So there is a sentence ϕ_w for each w which is true only at w. Take n of these ϕ_w sentences – none is strictly equivalent. So obviously $n-1$ of them are not strictly equivalent.

   (b) If for any 5 sentences, at least 4 must be strictly equivalent, then there can be no more than 2 expressible propositions – if there were 3 propositions expressible, we could have strictly inequivalent sentences expressing each of those propositions as part of our 5 sentences, so that at most 3 of our 5 would be strictly equivalent. If there can be no more than 2 expressible propositions, we cannot have more than 1 world in a $\mathscr{L}_{\mathsf{K}}$ model. (If there were two worlds, we would at least have these strictly inequivalent sentences: $p \wedge \neg p$, $p \vee \neg p$, p, $\neg p$.)

6. **(a)** Yes an extension, yes a proper extension: there is a model of $\mathscr{L}^{t}_{\mathsf{K}}$ in which $\Box p \to p$ is false at some world, but that sentence is valid in $\mathscr{L}^{rt}_{\mathsf{K}}$.

   (b) Not an extension, by the previous answer.

   (c) $\Box A \to \Box\Box A$ is valid in all models of $\mathscr{L}^{t}_{\mathsf{K}}$. Suppose there was one model in which $v(w, \Box A \to \Box\Box A) = 0$. Then $v(w, \Box A) = 1$ and $v(w, \Box\Box A) = 0$. Then there is a w' such that $w\mathscr{R}w'$ and $v(w', \Box A) = 0$; and there is a w'' such that $w'\mathscr{R}w''$ and $v(w'', A) = 0$. But by transitivity, $w\mathscr{R}w''$, so that $v(w'', A) = 1$. Contradiction: there is no such model.

   (d) It suffices to show that there is a $\mathscr{L}^{rt}_{\mathsf{K}}$ model in which it fails (because those are also $\mathscr{L}^{t}_{\mathsf{K}}$ models). The one world model where $\mathscr{R} = \{\langle w, w, \rangle\}$ and where $v(w, A) = 0$ suffices: then $v(w, \Diamond A) = 0$ and then $v(w, \Diamond\Diamond A) = 0$. This is (trivially) transitive and reflexive.

   (e) Consider the model with domain $\{w_0, w_1, w_2, \ldots\}$, such that $w_i\mathscr{R}w_j$ whenever $j > i$. Let $v(w_1, p) = 1$ and $v(w_i, p) = 0$ for $i > 1$. Then $v(w_0, \Diamond p) = 1$, but since every world accesses a world in which p is false (by transitivity), $\Box p$ is false at every world, and so there is no world accessible from w_0 at which $\Box p$ is true, so $v(w_0, \Diamond\Box p) = 0$. The counterexample is infinite because – without reflexivity – we need infinitely many worlds to ensure that every world has a successor under $\mathscr{R}$ to avoid $\Box p$ being degenerately true at a world.